COHOMOLOGY AND THE CLASSIFICATION OF LIFTINGS

BY
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1. **Introduction.** In this paper we will be concerned with the problem of homotopy classification of liftings of a map. Suppose that $\beta=(E, B, p)$ is a locally trivial fibre space with fibre F. Then, for $f\colon X\to B$, there is the set $L(X,f,\beta)$ of homotopy classes of liftings of f. Assuming that $L(X,f,\beta)$ is not empty and that X is $(2n-1)$-coconnected and F is $(n-1)$-connected, we will construct, for $\alpha\in L(X,f,\beta)$, an abelian group structure on $L(X,f,\beta)$ such that α is the zero element. Denote the group by $(L(X,f,\beta),\alpha)$ and the sum of two elements γ_1 and γ_2 by $\gamma_1+_\alpha\gamma_2$.

Given $\alpha_0,\alpha_1\in L(X,f,\beta)$, the different group structures on $L(X,f,\beta)$ determined by α_0 and α_1 are isomorphic in the best possible way. The translation $(L(X,f,\beta),\alpha_1)\to(L(X,f,\beta),\alpha_0)$ which sends γ to $\gamma+_{\alpha_1}\alpha_0$ is an isomorphism.

A weak form of the classification problem is then to determine the structure of the group $(L(X,f,\beta),\alpha)$.

In §2 we define a B-cohomology theory where B is a fixed space. These are generalizations of cohomology with local coefficients, and if B is a point, they are generalized cohomology theories as in [9]. For each CW-pair (X, A) and integer n, $h^n(X, A)$ is a local system of abelian groups over the mapping space $\mathscr{M}(X, B)$. The group assigned to $f\in\mathscr{M}(X, B)$ is denoted by $h^n(X, A, f)$.

In §4 we construct the spectral sequence for a fibre map $\pi\colon Y\to X$. This is analogous to Dold's generalization of the Serre spectral sequence [2].

In §§5 and 6 we define a B-spectrum and show how to construct a B-cohomology theory from a B-spectrum. We then associate to a fibre space $\beta=(E, B, p)$ a B-spectrum $\mathscr{S}(\beta)$ in a natural way and define, for $\alpha\in L(X,f,\beta)$, a correspondence

$$\psi_\alpha\colon L(X,f,\beta)\to h^0(X,f,\mathscr{S}(\beta)),$$

which, in the stable range, i.e., when X is $(2n-1)$-coconnected and F is $(n-1)$-connected, is one-one and onto. The group structure on $L(X,f,\beta)$ having α as zero element is obtained by pulling back the group structure on $h^0(X,f,\mathscr{S}(\beta))$ via ψ_α. Then (using the terminology of §7) we show that $L(X,f,\beta)$ has a natural affine group structure.

Suppose that G is a finite group which acts on X and Y and is free on X. Let $E(X, Y)$ denote the set of homotopy classes of equivariant maps from X to Y. Using a construction of Heller [3] and our previous results, we define an affine

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group structure on $E(X, Y)$, provided $E(X, Y)$ is not empty, X/G is $(2n-1)$-coconnected and Y is $(n-1)$-connected. Our main result on the structure of $(E(X, Y), \alpha)$ is Theorem (8.13).

In §§9 and 10 we make use of recent results of Hirsch and Haefliger [4], [5] which reduce the problem of classifying immersions (embeddings) of a closed n-dimensional C^∞-manifold M in euclidean space E^{n+k} to a problem of classifying equivariant maps. These allow us to define a natural affine group structure on the set $IM^{n+k}(M)(EM^{n+k}(M))$ of regular homotopy classes of immersions (isotopy classes of embeddings) of M in E^{n+k} provided the set is not empty and $2k>n+1$ $(2k>n+3)$. Using Theorem (8.13) we compute the rank and p-primary component, p-odd, of these groups.

In §11 we study a question raised by Lashof and Smale [7] as to what classes in $H^k(M)$ are realizable as normal classes of an immersion of M into E^{n+k}.

2. **Cohomology theories.** Given a space X, let $\bar{X}$ denote the category whose objects are points of X and such that the set of maps $M(x_0, x_1)$, $x_0, x_1 \in X$, consists of equivalence classes of paths from x_1 to x_0, the equivalence relation being homotopy relative to the end points. A continuous map $f\colon X_1 \to X_2$ defines a covariant functor $\bar{f}\colon \bar{X}_1 \to \bar{X}_2$ in the obvious way.

Let $\mathscr{A}$ denote the category of abelian groups. A *local system* of abelian groups over X is a covariant functor $L\colon \bar{X} \to \mathscr{A}$. We will denote $L([\sigma])$ by $\sigma_\#$ where $[\sigma]$ is an equivalence class of paths.

Suppose that local systems $L_1\colon \bar{X}_1 \to \mathscr{A}$ and $L_2\colon \bar{X}_2 \to \mathscr{A}$ and a map $f\colon X_1 \to X_2$ are given. A *homomorphism* ψ over f from L_1 to L_2 is a natural transformation $\psi\colon L_1 \to L_2\bar{f}$.

Let $\mathscr{L}$ denote the category whose objects are pairs (X, L) where L is a local system over X and whose maps are pairs $(f, \psi)\colon (X_1, L_1) \to (X_2, L_2)$, where $f\colon X_1 \to X_2$ and ψ is a homomorphism over f from L_1 to L_2.

Let $\mathscr{P}^2$ denote the category of CW-pairs. Fix a space B. For any space X let $\mathscr{M}(X, B)$ denote the space with the compact-open topology of maps $f\colon X \to B$. For $g\colon X_1 \to X_2$ define $\mathscr{M}(g)\colon \mathscr{M}(X_2, B) \to \mathscr{M}(X_1, B)$ by $\mathscr{M}(g)(f) = fg$.

A *B-cohomology theory on* $\mathscr{P}^2$ consists of the following.

(A). For $(X, A) \in \mathscr{P}^2$, $f \in \mathscr{M}(X, B)$ and each integer n, an abelian group $h^n(X, A, f)$.

(B). For $(X, A) \in \mathscr{P}^2$ and $F\colon I \to \mathscr{M}(X, B)$, with $F(0)=f_0$, $F(1)=f_1$, a homomorphism

$$F_\#\colon h^n(X, A, f_1) \to h^n(X, A, f_0).$$

(C). For $g\colon (X_1, A_1) \to (X_2, A_2)$ and $f \in \mathscr{M}(X_2, B)$, a homomorphism

$$g^*\colon h^n(X_2, A_2, f) \to h^n(X_1, A_1, fg).$$

(D). For $(X, A) \in \mathscr{P}^2$ and $f \in \mathscr{M}(X, B)$ a homomorphism

$$d\colon h^n(A, f|_A) \to h^{n+1}(X, A, f).$$

These are to have the following properties.

I. For $(X, A) \in \mathscr{P}^2$, the collection $\{h^n(X, A, f), F_\#\}$, $f \in \mathscr{M}(X, B)$, $F \in \mathscr{M}(X, B)^I$, is a local system over $\mathscr{M}(X, B)$ which will be denoted by $h^n(X, A)$.

II. For $g: (X_1, A_1) \to (X_2, A_2)$ the collection $\{g^*: h^n(X_2, A_2, f) \to h^n(X_1, A_1, fg)\}$, $f \in \mathscr{M}(X_2, B)$ is a homomorphism of local systems over $\mathscr{M}(g)$.

Then, for $(X, A) \in \mathscr{P}^2$, the collection $\{h^n(A, f|_A), F|_{A\#}\}, f \in \mathscr{M}(X, B)$, $F \in \mathscr{M}(X, B)^I$ is a local system over $\mathscr{M}(X, B)$. Here $F|_A: I \to \mathscr{M}(A, B)$ is defined by $F|_A(t)(a) = F(t)(a)$, $a \in A$.

III. For $(X, A) \in \mathscr{P}^2$, the collection $\{d: h^n(A, f|_A) \to h^{n+1}(X, A, f)\}, f \in \mathscr{M}(X, B)$ is a homomorphism of local systems over the identity map $\mathscr{M}(X, B) \to \mathscr{M}(X, B)$.

IV. The function $\mathscr{H}^n: \mathscr{P}^2 \to \mathscr{L}$ defined by $\mathscr{H}^n(X, A) = (\mathscr{M}(X, B), h^n(X, A))$ and $\mathscr{H}^n(g) = (\mathscr{M}(g), g^*)$ is a contravariant functor.

V. For $g: (X_1, A_1) \to (X_2, A_2)$ and $f \in \mathscr{M}(X_2, B)$, the diagram

$$\begin{tikzcd} h^n(A_2, f|_{A_2}) \arrow[r, "d"] \arrow[d, "(g|_{A_1})^*"] & h^{n+1}(X_2, A_2, f) \arrow[d, "g^*"] \\ h^n(A_1, f|_{A_2} g|_{A_1}) \arrow[r, "d"] & h^{n+1}(X_1, A_1, fg) \end{tikzcd}$$

is commutative.

VI. For $G: (X_1, A_1) \times I \to (X_2, A_2)$ a homotopy from g_0 to g_1, the diagram

$$\begin{tikzcd} & h^n(X_1, A_1, fg_1) \arrow[d, "(fG)_\#"] \\ h^n(X_2, A_2, f) \arrow[ur, "g_1^*"] \arrow[dr, "g_0^*"'] & \\ & h^n(X_1, A_1, fg_0) \end{tikzcd}$$

is commutative.

VII. For $(X, A) \in \mathscr{P}^2$ and $f \in \mathscr{M}(X, B)$, the sequence

$$\cdots \xrightarrow{i^*} h^n(A, f|_A) \xrightarrow{d} h^{n+1}(X, A, f) \xrightarrow{j^*} h^{n+1}(X, f) \xrightarrow{i^*} \cdots$$

is exact. Here $j: X \to (X, A)$ and $i: A \to X$ are inclusions.

VIII. If $X = A_1 \cup A_2$ and $(A_1, A_1 \cap A_2)$ and (X, A_2) are in $\mathscr{P}^2$, then, for $f \in \mathscr{M}(X, B)$,

$$i^*: h^n(X, A_2, f) \to h^n(A_1, A_1 \cap A_2, f|_{A_1})$$

is an isomorphism. Here $i: (A_1, A_1 \cap A_2) \to (X, A_2)$ is inclusion.

EXAMPLE 1. Take B to be a point. Then any generalized cohomology theory (such as in [9]) may be regarded as a B-cohomology theory on $\mathscr{P}^2$.

For the next example, if L is a local system of abelian groups over X, let $H^n(X, A; L)$ denote the nth singular cohomology group of (X, A) with coefficients in L.

EXAMPLE 2. Fix a space B and a local system L over B. For $(X, A) \in \mathscr{P}^2$ and

$f \in \mathcal{M}(X, B)$, the composition $L\bar{f}: \bar{X} \to \mathscr{A}$ is a local system over X. We then have $H^n(X, A; L\bar{f})$.

For $F: I \to \mathcal{M}(X, B)$ with $F(0)=f_0$, $F(1)=f_1$, and $x \in X$, define $\sigma_x: I \to B$ by $\sigma_x(t)=F(x, t)$, $0 \leqq t \leqq 1$. Then let $F_{\#}=\sigma_{x\#}: L(f_1(x)) \to L(f_0(x))$. There results a coefficient homomorphism

$$F_{\#}: H^n(X, A, L\bar{f}_1) \to H^n(X, A, L\bar{f}_0).$$

With the homomorphism induced by a continuous mapping of pairs and the boundary operator defined in the usual way, it is easy to see that we have a B-cohomology theory on $\mathscr{P}^2$.

3. **Description of $F_{\#}$.** Assume that a cohomology theory over B is given. Axioms IV and VI imply.

(3.1) Lemma. *If $g: (X_1, A_1) \to (X_2, A_2)$ is a homotopy equivalence, then for $f \in \mathcal{M}(X_2, B)$,*

$$g^*: h^n(X_2, A_2, f) \to h^n(X_1, A_1, fg)$$

is an isomorphism.

A subspace $A \subset X$ is a weak deformation retract of X if the inclusion $i: A \to X$ is a homotopy equivalence. By the exact cohomology sequence for (X, A) and the above lemma, we have

(3.2) Lemma. *If A is a weak deformation retract of X, then for $f \in \mathcal{M}(X, B)$, $h^n(X, A, f)=0$.*

Now let $f_0, f_1 \in \mathcal{M}(X, B)$ and let $F: X \times I \to B$ be a homotopy from f_0 to f_1. Let $\dot{I}=\{0, 1\}$. We then have a boundary operator

$$d_j: h^n(X \times \dot{I} \cup A \times I, X \times \{j\} \cup A \times I, F) \to h^{n+1}(X \times I, X \times \dot{I} \cup A \times I, F), \quad j = 0, 1. \tag{3.3}$$

(3.4) Lemma. *For $j=0, 1$, d_j is an isomorphism.*

Proof. By exactness, it is sufficient to show that $h^n(X \times I, X \times \{j\} \cup A \times I, F)=0$. This follows from the preceding lemma.

Let $\varepsilon(j)=0$ if $j=1$ and $\varepsilon(j)=1$ if $j=0$. Define $i_j: (X, A) \to (X \times \dot{I} \cup A \times I, X \times \{\varepsilon(j)\} \cup A \times I)$ by $i_j(x)=(x, j)$, $j=0, 1$. By Axiom VIII,

$$i_j^*: h^n(X \times \dot{I} \cup A \times I, X \times \{\varepsilon(j)\} \cup A \times I, F) \to (X, A, f_j) \tag{3.5}$$

is an isomorphism. Hence we have a suspension isomorphism

$$s_j: h^{n+1}(X \times I, X \times \dot{I} \cup A \times I, F) \to h^n(X, A, f_{\varepsilon(j)}), \quad j = 0, 1, \tag{3.6}$$

by $s_j=i_{\varepsilon(j)}^* d_j^{-1}$.

Let $\pi: X \times I \to X$ be the projection.

(3.7) LEMMA. *For $f \in \mathscr{M}(X, B)$, the composition*

$$h^n(X, A, f) \xrightarrow{s_1^{-1}} h^{n+1}(X\times I, X\times \dot{I} \cup A\times I, f\pi) \xrightarrow{s_0} h^n(X, A, f)$$

is minus the identity.

The proof is the same as for ordinary cohomology.

The next theorem characterizes $F_{\#}$ in terms of suspension.

(3.8) THEOREM. *Let $f_0, f_1 \in \mathscr{M}(X, B)$ and let $F\colon X\times I \to B$ be a homotopy from f_0 to f_1. Commutativity holds in the diagram*

$$\begin{tikzcd}[column sep=small]
 & h^n(X, A, f_1) \arrow[d, "-F_{\#}"] \\
h^{n+1}(X\times I, X\times \dot{I} \cup A\times I, F) \arrow[ur, "s_0"] \arrow[r, "s_1"'] & h^n(X, A, f_0)
\end{tikzcd}$$

Proof. Define $M\colon (X\times I)\times I \to X\times I$ by $M(x, t)(\lambda) = (x, t\lambda)$. Then $FM_0 = f_0\pi$ and $FM_1 = F$. We have by Axioms II and III a commutative diagram

$$\begin{tikzcd}
h^n(X, A, f_1) \arrow[r, "(\mathscr{M}(i_1)FM)_{\#}"] & h^n(X, A, f_0) \\
h^{n+1}(X\times I, X\times \dot{I} \cup A\times I, F) \arrow[u, "s_0"] \arrow[r, "(FM)_{\#}"] \arrow[d, "s_1"] & h^{n+1}(X\times I, X\times \dot{I} \cup A\times I, F) \arrow[u, "s_0"'] \arrow[d, "s_1"'] \\
h^n(X, A, f_0) \arrow[r, "(\mathscr{M}(i_0)FM)_{\#}"] & h^n(X, A, f_0)
\end{tikzcd} \tag{3.9}$$

By the previous lemma $s_1 s_0^{-1}$ on the right is minus the identity. Next, we have $\mathscr{M}(i_1)FM = F$ in $\mathscr{M}(X, B)^I$ and $\mathscr{M}(i_0)FM$ in $\mathscr{M}(X, B)^I$ is the constant path on f_0. Therefore $(\mathscr{M}(i_1)FM)_{\#} = F_{\#}$ and $(\mathscr{M}(i_0)FM)_{\#}$ is the identity. It follows that $s_1 s_0^{-1}$ on the left is $-F_{\#}$.

Let $\tau_p = \langle v_0 \cdots v_p\rangle$ be an euclidean p-simplex and $\dot{\tau}_p$ its boundary. Let $\tau_{p,i} = \langle v_0 \cdots v_{i-1}, v_{i+1} \cdots v_p\rangle$ and let $J_i(\tau_p)$ be the closure of $\dot{\tau}_p - \tau_{p,i}$, $0 \leqq i \leqq p$.

For $F \in \mathscr{M}(X\times \tau_p, B)$, define

$$s_i\colon h^n(X\times \tau_p, X\times \dot{\tau}_p, F) \to h^{n-1}(X\times \tau_{p,i}, X\times \dot{\tau}_{p,i}, F), \tag{3.10}$$

$0 \leqq i \leqq p$, to be the composition

$$h^n(X\times \tau_p, X\times \dot{\tau}_p, F) \xrightarrow{d^{-1}} h^{n-1}(X\times \dot{\tau}_p, X\times J_i(\tau_p), F)$$
$$\xrightarrow{i^*} h^{n-1}(X\times \tau_{p,i}, X\times \dot{\tau}_{p,i}, F).$$

Applying s_0 p-times, we obtain

$$s_0^p\colon h^n(X\times \dot{\tau}_p, X\times \dot{\tau}_p, F) \to h^{n-p}(X\times \{v_p\}, F). \tag{3.11}$$

Define $\lambda_k\colon X \to X \times \tau_p$ by $\lambda_k(x) = (x, v_k)$, $k = p-1, p$, and let $\pi\colon X \times \tau_p \to X$ be the projection.

(3.12) Lemma. *For $f \in \mathscr{M}(X, B)$, the diagram*

$$\begin{tikzcd} h^n(X \times \tau_p, X \times \dot\tau_p, f\pi) \arrow[r, "s_i"] \arrow[d, "\lambda_p^* s_0^p"] & h^n(X \times \tau_{p,i}, X \times \dot\tau_{p,i}, f\pi) \arrow[d, "\lambda_p^* s_0^{p-1}"] \\ h^{n-p}(X, f) \arrow[r, "(-1)^i"] & h^{n-p}(X, f) \end{tikzcd}$$

is commutative, where $k = p$ if $0 \leqq i \leqq p$, and $k = p-1$ if $i = p$.

The proof is the same as for ordinary cohomology.

Now for $F \in \mathscr{M}(X \times \tau_p, B)$ consider the diagrams

$$\begin{tikzcd} h^n(X \times \tau_p, X \times \dot\tau_p, F) \arrow[r, "s_i"] \arrow[d, "\lambda_p^* s_0^p"] & h^n(X \times \tau_{p,i}, X \times \dot\tau_{p,i}, F) \arrow[d, "\lambda_p^* s_0^{p-1}"] \\ h^n(X, F\lambda_p) \arrow[r, "(-1)^i"] & h^n(X, F\lambda_p), \end{tikzcd} \qquad 0 \leqq i \leqq p \tag{3.13}$$

$$\begin{tikzcd} h^n(X \times \tau_p, X \times \dot\tau_p, F) \arrow[r, "s_p"] \arrow[d, "\lambda_p^* s_0^p"] & h^n(X \times \tau_{p,p}, X \times \dot\tau_{p,p}, F) \arrow[d, "\lambda_{p-1}^* s_0^{p-1}"] \\ h^n(X, F\lambda_p) \arrow[r, "(-1)^p T_{\#}"] & h^n(X, F\lambda_{p-1}) \end{tikzcd} \tag{3.14}$$

where $T\colon X \times I \to B$ is defined by $T(x, t) = F(x, tv_{p-1} + (1-t)v_p)$, $0 \leqq t \leqq 1$.

(3.15) Lemma. *The diagrams* (3.13) *and* (3.14) *are commutative.*

The proof is similar to the proof of Theorem (3.8). Here we use the preceding lemma, the homotopy $M\colon (X \times \tau_p) \times I \to X \times \tau_p$ by $M(x, z)(\lambda) = \lambda z + (1-\lambda)v_p$, $0 \leqq \lambda \leqq 1$, and a diagram similar to (3.9).

4. **The spectral sequence.** Assume that a space B and a B-cohomology theory on $\mathscr{P}^2$ is given. In this section we construct the spectral sequence associated with a fibre map $\pi\colon Y \to X$. This is a generalization of the Serre-Dold spectral sequence [2].

We assume that $\pi\colon Y \to X$ is locally trivial, X is a polyhedron and for each pair (K, L) of subcomplexes of X, we have $(\pi^{-1}(K), \pi^{-1}(L)) \in \mathscr{P}^2$.

Let $f \in \mathscr{M}(X, B)$ be given. Let $F(x) = \pi^{-1}(x)$, $x \in X$. We now describe the way in which the collection of groups $h^n(F(x), f\pi)$, $x \in X$, is a local system over X. Let $\sigma\colon I \to X$ be a path from x_0 to x_1. By the covering homotopy property, there is $S(\sigma)\colon F(x_0) \times I \to X$ such that $S(\sigma)$ covers σ and $S(\sigma)_0\colon F(x_0) \to F(x_0)$ is the identity. We then have $S(\sigma)_1\colon F(x_0) \to F(x_1)$.

Next, for $x \in X$, we have $T(x, \sigma)\colon I \to \mathcal{M}(F(x), B)$ by $T(x, \sigma)(t)(y) = f\sigma(t)$, $0 \leqq t \leqq 1$, $y \in F(x)$. Note that $T(x_0, \sigma)(1) = f\pi S(\sigma)_1$. Let

$$\sigma_{\#}\colon h^n(F(x_1), f\pi) \to h^n(F(x_0), f\pi) \tag{4.1}$$

be the composition

$$h^n(F(x_1), f\pi) \xrightarrow{S(\sigma)_1^*} h^n(F(x_0), f\pi S(\sigma)_1) \xrightarrow{T(x_0, \sigma)_{\#}} h^n(F(x_0), f\pi).$$

(4.2) Lemma. *The assignment of $h^n(F(x), f\pi)$ to $x \in X$ and of $\sigma_{\#}$ to $\sigma \in X^I$ is a local system on X.*

Proof. Let σ be a path from x_0 to x_1 and τ a path from x_1 to x_2. We will show that $(\sigma\tau)_{\#} = \sigma_{\#}\tau_{\#}$ and leave the other properties to the reader. Consider the diagram

$$\begin{tikzcd}
 & h^n(F(x_1), f\pi S(\tau)_1) \arrow[rr, "T(x_1, \tau)_{\#}"] \arrow[dd, "S(\sigma)_1^*"] & & h^n(F(x_1), f\pi) \arrow[dd, "S(\sigma)_1^*"] \\
h^n(F(x_2), f\pi) \arrow[ur, "S(\tau)_1^*"] \arrow[dr, "S(\sigma\tau)_1^*"'] & & & \\
 & h^n(F(x_0), f\pi S(\sigma\tau)_1) \arrow[rr, "T(x_0, \tau)_{\#}"] \arrow[dr, "T(x_0, \sigma\tau)_{\#}"'] & & h^n(F(x_0), f\pi S(\sigma)_1) \arrow[dl, "T(x_0, \sigma)_{\#}"] \\
 & & h^n(F(x_0), f\pi) &
\end{tikzcd}$$

The left hand triangle is commutative, since we may take $S(\sigma\tau)_1$ to be the composition $S(\tau)_1 S(\sigma)_1$. The lower triangle is commutative by Axiom I. The square is commutative by Axiom II. Thus $(\sigma\tau)_{\#} = \sigma_{\#}\tau_{\#}$.

We will denote the local system described above by $[h^n(F)]$.

Let X_p be the p-skeleton of X and let $Y_p = \pi^{-1}(X_p)$. We have an exact sequence

$$\cdots \xrightarrow{d} h^n(Y, Y_p, f\pi) \xrightarrow{i^*} h^n(Y, Y_{p-1}, f\pi) \xrightarrow{j^*} h^n(Y_p, Y_{p-1}, f\pi) \xrightarrow{d} \cdots.$$

Piecing these together leads to an exact couple with

$$E_1^{p,q} = h^{p+q}(Y_p, Y_{p-1}, f\pi), \qquad D_1^{p,q} = h^{p+q}(Y, Y_{p-1}, f\pi). \tag{4.3}$$

Fix a total ordering of the vertices of X. For $\tau_p = \langle v_0 \cdots v_p \rangle$, define

$$\tilde{s}_i\colon h^{p+q}(\pi^{-1}(\tau_p), \pi^{-1}(\dot{\tau}_p), f\pi) \to h^{p+q-1}(\pi^{-1}(\tau_{p,i}), \pi^{-1}(\dot{\tau}_{p,i}), f\pi) \tag{4.4}$$

to be i^*d^{-1}, where d is from the cohomology sequence of the triple $(\pi^{-1}(\tau_p), \pi^{-1}(\tau_{p,i}), \pi^{-1}(J_i(\tau_p)))$ and $i\colon (\pi^{-1}(\tau_{p,i}), \pi^{-1}(\dot{\tau}_{p,i})) \to (\pi^{-1}(\dot{\tau}_p), \pi^{-1}(J_i(\tau_p)))$ is the inclusion.

Applying $\tilde{s}_0$ p-times leads to an isomorphism

$$\tilde{s}_0^p\colon h^{p+q}(\pi^{-1}(\tau_p), \pi^{-1}(\dot{\tau}_p), f\pi \to h^q(F(v_p), f\pi). \tag{4.5}$$

Consider the diagrams

$$
\begin{tikzcd}
h^{p+q}(\pi^{-1}(\tau_p), \pi^{-1}(\dot{\tau}_p), f\pi) \arrow[r, "\tilde{s}_i"] \arrow[d, "\tilde{s}_0^p"] & h^{p+q-1}(\pi^{-1}(\tau_{p,i}), \pi^{-1}(\dot{\tau}_{p,i}), f\pi) \arrow[d, "\tilde{s}_0^{p-1}"] \\
h^q(F(v_p), f\pi) \arrow[r, "(-1)^i"] & h^q(F(v_p), f\pi), \quad 0 \leqq i < p,
\end{tikzcd}
\tag{4.6}
$$

$$
\begin{tikzcd}
h^{p+q}(\pi^{-1}(\tau_p), \pi^{-1}(\dot{\tau}_p), f\pi) \arrow[r, "\tilde{s}_p"] \arrow[d, "\tilde{s}_0^p"] & h^{p+q-1}(\pi^{-1}(\tau_{p,p}), \pi^{-1}(\dot{\tau}_{p,p}), f\pi) \arrow[d, "\tilde{s}_0^{p-1}"] \\
h^q(F(v_p), f\pi) \arrow[r, "(-1)^p\sigma_{\#}"] & h^q(F(v_{p-1}), f\pi)
\end{tikzcd}
\tag{4.7}
$$

where $\sigma: I \to X$ is defined by $\sigma(t) = tv_p + (1-t)v_{p-1}$, $0 \leqq t \leqq 1$.

(4.8) LEMMA. *The diagrams* (4.6) *and* (4.7) *are commutative.*

Proof. We will show that (4.7) is commutative. Choose $S: F(v_{p-1}) \times \tau_p \to \pi^{-1}(\tau_p)$ to cover the inclusion $\tau_p \subset X$ and such that $S\lambda_{p-1}: F(v_{p-1}) \to F(v_{p-1})$ is the identity. We have a commutative diagram

$$
\begin{tikzcd}
h^q(F(v_{p-1}), f\pi S\lambda_p) & h^q(F(v_p), f\pi) \arrow[l, "(S\lambda_p)^*"'] \\
h^{p+q}(F(v_{p-1}) \times \tau_p, F(v_{p-1}) \times \dot{\tau}_p, f\pi S) \arrow[u, "\lambda_p^* s_0^p"] \arrow[d, "\lambda_{p-1}^* s_0^{p-1} s_p"'] & h^{p+q}(\pi^{-1}(\tau_p), \pi^{-1}(\dot{\tau}_p), f\pi) \arrow[l, "S^*"'] \arrow[u, "\tilde{s}_0^p"'] \arrow[d, "\tilde{s}_0^{p-1}\tilde{s}_p"] \\
h^q(F(v_{p-1}), f\pi S\lambda_{p-1}) & h^q(F(v_{p-1}), f\pi) \arrow[l, "(S\lambda_{p-1})^*"']
\end{tikzcd}
$$

Now use this, the commutativity of (3.14) and the fact that $S\lambda_{p-1}$ is the identity to deduce that

$$\tilde{s}_0^{p-1}\tilde{s}_p = (-1)^p T_{\#}(S\lambda_p)^* \tilde{s}_0^p.$$

Next, note that $T = T(v_{p-1}, \sigma)$. Therefore $T_{\#}(S\lambda_p)^* = \sigma_{\#}$. The proof that (4.6) is commutative is similar.

For $\tau_p \subset X$, let $i(\tau_p): \pi^{-1}(\tau_p) \to Y_p$ be the inclusion. We have

$$i(\tau_p)^*: h^{p+q}(Y_p, Y_{p-1}, f\pi) \to h^{p+q}(\pi^{-1}(\tau_p), \pi^{-1}(\dot{\tau}_p), f\pi). \tag{4.9}$$

Let $C^*(X; [h^q(F)])$ denote the simplicial cochain complex of X with coefficients in $[h^q(F)]$. Define

$$\psi: h^{p+q}(Y_p, Y_{p-1}, f\pi) \to C^p(X; [h^q(F)]) \tag{4.10}$$

by $\psi(u)(\tau_p) = \tilde{s}_0^p i(\tau_p)^*(u)$, $u \in h^{p+q}(Y_p, Y_{p-1}, f\pi)$. Then ψ is an isomorphism and, by Lemma (4.8), commutes with the boundary operator. Therefore we have an identification

$$\psi: E_2^{p,q} \to H^p(X; [h^q(F)]). \tag{4.11}$$

We have a filtration

$$h^n(Y, f\pi) = J^{0,n} \supset \cdots \supset J^{p,n-p} \supset \cdots, \tag{4.12}$$

where

$$J^{p,n-p} = \text{Image}\,(h^n(Y, Y_{p-1}, f\pi) \to h^n(Y, f\pi)).$$

As usual, let $E_\infty^{p,n-p} = J^{p,n-p}/J^{p+1,n-p-1}$. We will discuss now the convergence of $\{E_r, d_r\}$ to E_∞.

Definition. A pair (X, A) is *k-coconnected* if for every local system L of abelian groups over X, we have $H^q(X, A; L) = 0$, $q \geqq k$.

(4.13) Lemma. *If Y is k-coconnected and $F(x)$, $x \in X$, is l-coconnected, then* $D_r^{p,q} = 0$, $r > \max(k+2-p, l+1)$.

Proof. By inspecting the singular cohomology spectral sequence of $\pi\colon Y_s \to Y_s$, we see that Y_s is $(s+l)$-coconnected. Therefore (Y, Y_{p-1}) is s-coconnected if $s > \max(k, p-1+l)$. Now take $r > \max(k+2-p, l+1)$. Then $p+r-2 > \max(k, p-1+l)$ so that by obstruction theory, there is $M\colon Y \times I \to Y$ such that M_0 is the identity, $M_1(Y) \subset Y_{p+r-2}$, and M_t restricted to Y_{p-1} is the identity, $0 \leqq t \leqq 1$. We have a commutative diagram

$$\begin{tikzcd}
h^{p+q}(Y, Y_{p-1}, f\pi) \arrow[r, "i^*"] \arrow[drr, "M_0^*"'] & h^{p+q}(Y_{p+r-2}, Y_{p-1}, f\pi) \arrow[r, "M_1^*"] & h^{p+q}(Y, Y_{p-1}, f\pi M_1) \arrow[d, "(f\pi M)_\#"] \\
 & & h^{p+q}(Y, Y_{p-1}, f\pi)
\end{tikzcd}$$

which implies that i^* is injective. By exactness,

$$D_r^{p,q} = \text{Image}\,(h^{p+q}(Y, Y_{p+r-2}, f\pi) \to h^{p+q}(Y, Y_{p-1}, f\pi)) = 0.$$

(4.14) Theorem. *Suppose that either* (a) *X is finitely coconnected or* (b) *Y and $F(x)$, $x \in X$, are finitely coconnected. Then*

(1) *For each pair (p, q), there is an integer $r(p, q)$ such that* $E_{r(p,q)}^{p,q} \simeq E_\infty^{p,q}$.

(2) *The filtration* (4.12) *is finite.*

This follows by a standard spectral sequence argument. For case (b), the preceding lemma is needed.

5. **Liftings.** Suppose that we have a pair (β, Δ) where $\beta = (E, B, p)$ is a Serre fibre space and $\Delta\colon B \to E$ is a cross-section ($p\Delta =$ identity). For $(X, A) \in \mathscr{P}^2$, let

$$\mathscr{L}(X, A, \beta, \Delta) = \{g\colon X \to E \mid g_{|A} = \Delta p g_{|A}\} \tag{5.1}$$

and define

$$\omega\colon \mathscr{L}(X, A, \beta, \Delta) \to \mathscr{M}(X, B) \tag{5.2}$$

by $\omega(g) = pg$.

(5.3) LEMMA. *The map ω is a Serre fibre map.*

This follows easily from the exponential law and the fact that p is a Serre fibre map.

Note that ω has a cross-section

$$\delta: \mathscr{M}(X, B) \to \mathscr{L}(X, A, \beta, \Delta) \tag{5.4}$$

defined by $\delta(f) = \Delta f$.

The fibre above $f \in \mathscr{M}(X, B)$ will be denoted by $\mathscr{L}(X, A, f, \beta, \Delta)$. When we speak of the homotopy groups of $\mathscr{L}(X, A, f, \beta, \Delta)$, it will be understood that the base-point is Δf.

Let $F: I \to \mathscr{M}(X, B)$ be a path from f_0 to f_1. Then δF is a path in $\mathscr{L}(X, A, \beta, \Delta)$ from Δf_0 to Δf_1. We have

$$(\delta F)_{\#}: \pi_n(\mathscr{L}(X, A, \beta, \Delta); \Delta f_1) \to \pi_n(\mathscr{L}(X, A, \beta, \Delta); \Delta f_0). \tag{5.5}$$

Now define

$$F_{\#}: \pi_n(\mathscr{L}(X, A, f_1, \beta, \Delta)) \to \pi_n(\mathscr{L}(X, A, f_0, \beta, \Delta)) \tag{5.6}$$

so that the diagram

$$\begin{tikzcd}
\pi_n(\mathscr{L}(X, A, f_1, \beta, \Delta)) \arrow[r, "F_{\#}"] \arrow[d, "i_{\#}"] & \pi_n(\mathscr{L}(X, A, f_0, \beta, \Delta)) \arrow[d, "i_{\#}"] \\
\pi_n(\mathscr{L}(X, A, \beta, \Delta); \Delta f_1) \arrow[r, "(\delta F)_{\#}"] & \pi_n(\mathscr{L}(X, A, \beta, \Delta); \Delta f_0)
\end{tikzcd}$$

is commutative. (This is possible because of the cross-section δ.) Then, as in [1], we have

(5.7) LEMMA. *The correspondence $f \to \pi_n(\mathscr{L}(X, A, f, \beta, \Delta))$ and $F \to F_{\#}$ is a local system on $\mathscr{M}(X, B)$.*

We consider the effect of a change of variable. For $g: (X_1, A_1) \to (X_2, A_2)$; we have a commutative diagram

$$\begin{tikzcd}
\mathscr{L}(X_2, A_2, \beta, \Delta) \arrow[r, "\mathscr{L}(g)"] \arrow[d, "\omega"] & \mathscr{L}(X_1, A_1, \beta, \Delta) \arrow[d, "\omega"] \\
\mathscr{M}(X_2, B) \arrow[r, "\mathscr{M}(g)"] & \mathscr{M}(X_1, B)
\end{tikzcd} \tag{5.8}$$

where $\mathscr{L}(g)(h) = hg$. Therefore the collection

$$\mathscr{L}(g)_{\#}: \pi_n(\mathscr{L}(X_2, A_2, f, \beta, \Delta)) \to \pi_n(\mathscr{L}(X_1, A_1, fg, \beta, \Delta)), \tag{5.9}$$

$f \in \mathscr{M}(X_2, B)$, is a homomorphism of local systems.

Now let (β_1, Δ_1) and (β_2, Δ_2) be given where $\beta_i = (E_i, B, p_i)$, $i = 1, 2$. By a map

$$(k, K): (\beta_1, \Delta_1) \to (\beta_2, \Delta_2) \tag{5.10}$$

we mean $k: E_1 \to E_2$ such that $p_2k=p_1$ and $K: B\times I \to E_2$ such that $K_0=\Delta_2$, $K_1=k\Delta_1$, and $p_2K_t=\text{identity}$, $0\leqq t\leqq 1$. Thus, up to the homotopy K, k is cross-section preserving. We have a commutative diagram

$$\begin{tikzcd} \mathscr{L}(X, A, \beta_1, \Delta_1) \arrow[rr, "\mathscr{L}(k)"] \arrow[dr, "\omega"'] & & \mathscr{L}(X, A, \beta_2, \Delta_2) \arrow[dl, "\omega"] \\ & \mathscr{M}(X, B) & \end{tikzcd} \tag{5.11}$$

where $\mathscr{L}(k)(h)=kh$. Note that for $f\in\mathscr{M}(X, B)$, the composition $I\xrightarrow{K}\mathscr{M}(B, E_2)$ $\xrightarrow{\mathscr{M}(f)}\mathscr{M}(X, E_2)$ is a path in $\mathscr{L}(X, A, \beta_2, \Delta_2)$ from $\Delta_2 f$ to $k\Delta_1 f$. Define

$$(k, K)_{\#}: \pi_n(\mathscr{L}(X, A, f, \beta_1, \Delta_1)) \to \pi_n(\mathscr{L}(X, A, f, \beta_2, \Delta_2)), \tag{5.12}$$

$f\in\mathscr{M}(X, B)$, to be the composition

$$\begin{tikzcd} \pi_n(\mathscr{L}(X, A, f, \beta_1, \Delta_1)) \arrow[r, "\mathscr{L}(k)_{\#}"] & \pi_n(\mathscr{L}(X, A, f, \beta_2, \Delta_2); k\Delta_1 f) \arrow[d, "(\mathscr{M}(f)K)_{\#}"] \\ & \pi_n(\mathscr{L}(X, A, f, \beta_2, \Delta_2)). \end{tikzcd}$$

It is easy to show that $(k, K)_{\#}$ is a homomorphism of local systems.

6. ***B*-spectra.** Given (β, Δ) as in the previous section, let

$$\Omega(E;\Delta) = \{\sigma: I\to E \mid \sigma(I)\subset p^{-1}(b), \text{ some } b\in B, \text{ and } \sigma(0)=\sigma(1)=\Delta(b)\} \tag{6.1}$$

and define

$$\Omega(p): \Omega(E;\Delta)\to B \tag{6.2}$$

by $\Omega(p)(\sigma)=b$, where $\sigma(I)\subset p^{-1}(b)$. Using the exponential law we see that $\Omega(\beta;\Delta) =(\Omega(E;\Delta), B, \Omega(p))$ is a Serre fibre space. Define a cross-section

$$\Omega(\Delta): B\to\Omega(E;\Delta) \tag{6.3}$$

by $\Omega(\Delta)(b)(t)=\Delta(b)$, $0\leqq t\leqq 1$.

The pair $(\Omega(\beta;\Delta), \Omega(\Delta))$ will be called the *loop space* of (β, Δ).

For $(X, A)\in\mathscr{P}^2$ and $f\in\mathscr{M}(X, B)$, the exponential law gives an identification

$$\mathscr{L}(X, A, f, \Omega(\beta;\Delta), \Omega(\Delta)) \to \Omega(\mathscr{L}(X, A, f, \beta, \Delta)). \tag{6.4}$$

This in turn leads to an identification

$$\pi_r(\mathscr{L}(X, A, f, \Omega(\beta;\Delta), \Omega(\Delta)) \to \pi_{r+1}(\mathscr{L}(X, A, f, \beta, \Delta)). \tag{6.5}$$

A *B-spectrum* $\mathscr{S}$ is a sequence of pairs (β_m, Δ_m), $-\infty<m<+\infty$, where $\beta_m =(E_m, B, p_m)$ is a Serre fibre space and $\Delta_m: B\to E_m$ is a cross-section, together with maps

$$(k_m, K_m): (\beta_m, \Delta_m)\to(\Omega(\beta_{m+1};\Delta_{m+1}), \Omega(\Delta_{m+1})). \tag{6.6}$$

Given a B-spectrum $\mathscr{S}$, we have for $(X, A) \in \mathscr{P}^2$ and $f \in \mathscr{M}(X, B)$, a homomorphism

$$(6.7) \qquad (k_m, K_m)_\#: \pi_n(\mathscr{L}(X, A, f, \beta_m, \Delta_m)) \to \pi_{n+1}(\mathscr{L}(X, A, f, \beta_{m+1}, \Delta_{m+1}))$$

(using the identification (6.5)). Now, for each integer n, let

$$(6.8) \qquad h^n(X, A, f; \mathscr{S}) = \operatorname{dir\,lim}_m \pi_{-n+m}(\mathscr{L}(X, A, f, \beta_{m+1}, \Delta_{m+1})).$$

Given $F: I \to \mathscr{M}(X, B)$ from f_0 to f_1, the homomorphisms

$$(6.9) \qquad F_\#: \pi_n(\mathscr{L}(X, A, f_1, \beta_m, \Delta_m)) \to \pi_n(\mathscr{L}(X, A, f_0, \beta_m, \Delta_m))$$

commute with those in (6.7). Let

$$(6.10) \qquad F_\#: h^n(X, A, f_1; \mathscr{S}) \to h^n(X, A, f_0; \mathscr{S})$$

be obtained from these by passing to the direct limit.

Given $g: (X_1, A_1) \to (X_2, A_2)$ and $f \in \mathscr{M}(X_2, B)$, the homomorphisms

$$(6.11) \qquad \mathscr{L}(g)_\#: \pi_n(\mathscr{L}(X_2, A_2, f, \beta_m, \Delta_m)) \to \pi_n(\mathscr{L}(X_1, A_1, fg, \beta_m, \Delta_m))$$

commute with those in (6.7). Define

$$(6.12) \qquad g^*: h^n(X_2, A_2, f; \mathscr{S}) \to h^n(X_1, A_1, fg; \mathscr{S})$$

to be the direct limit of the $\mathscr{L}(g)_\#$.

For $(X, A) \in \mathscr{P}^2$ let $i: A \to X$ and $j: X \to (X, A)$ be the inclusions. Then for $f \in \mathscr{M}(X, B)$ we have

$$(6.13) \qquad \mathscr{L}(X, A, f, \beta_m, \Delta_m) \xrightarrow{\mathscr{L}(j)} \mathscr{L}(X, f, \beta_m, \Delta_m) \xrightarrow{\mathscr{L}(i)} \mathscr{L}(A, f|_A, \beta_m, \Delta_m).$$

Using the exponential law we see that $\mathscr{L}(i)$ is a fibre map. Further, $\mathscr{L}(j)$ is an identification with the fibre $\mathscr{L}(i)^{-1}(\Delta_m f|_A)$. Therefore, we have an exact sequence

$$(6.14) \qquad \begin{aligned} \cdots &\xrightarrow{\mathscr{L}(i)_\#} \pi_n(\mathscr{L}(A, f|_A, \beta_m, \Delta_m)) \xrightarrow{\delta_\#} \pi_{n-1}(\mathscr{L}(X, A, f, \beta_m, \Delta_m)) \\ &\xrightarrow{\mathscr{L}(j)_\#} \pi_{n-1}(\mathscr{L}(X, f, \beta_m, \Delta_m)) \xrightarrow{\mathscr{L}(i)_\#} \cdots \end{aligned}$$

and the homomorphisms $\delta_\#$ commute with those in (6.7). Therefore we may define

$$(6.15) \qquad d: h^n(A, f|_A; \mathscr{S}) \to h^{n+1}(X, A, f; \mathscr{S})$$

to be the direct limit of the $\delta_\#$.

(6.16) THEOREM. *With h^n, $\#$, $*$, and d as defined in* (6.8), (6.10), (6.12) *and* (6.15) *we have a B-cohomology theory on $\mathscr{P}^2$.*

Proof. Using the results of §5, Axioms I through VI are easily checked. Axiom VII follows from the exactness of (6.14) and the fact that exactness is preserved under direct limit. For Axiom VIII, note that if $i: (A_1, A_1 \cap A_2) \to (X, A_2)$ is the inclusion, then for $f \in \mathcal{M}(X, B)$,

$$\mathcal{L}(i): \mathcal{L}(X, A_2, f, \beta_m, \Delta_m) \to \mathcal{L}(A_1, A_1 \cap A_2, f|_A, \beta_m, \Delta_m)$$

is a homeomorphism.

7. **The group structure.** The *suspension* $S(F)$ of a space F will be the quotient obtained from $F \times I$ by the identification

(7.1) $$(x, t) \sim (x', t) \quad \text{if and only if } x = x' \text{ or } t = 0 \text{ or } t = 1.$$

However, we will use a weaker topology than the usual one. Let $\omega: F \times I \to S(F)$ denote the projection. A basis for the topology on $S(F)$ is to consist of sets of the form $\omega(U \times (t_1, t_2))$, U open in F, $0 < t_1 < t_2 < 1$, or $\omega(F \times (t, 1])$, $t < 1$, or $\omega(F \times [0, t))$, $t > 0$.

Suppose that $\beta = (E, B, p)$ is a fibre space. Let $\Sigma(E)$ be the quotient obtained from $E \times I$ by the identification

(7.2) $$(e, t) \sim (e', t) \quad \text{if and only if } e = e' \text{ or } t = 0 \text{ or } 1 \text{ and } p(e) = p(e').$$

Let $\omega: E \times I \to \Sigma(E)$ be the projection. A basis for the topology on $\Sigma(E)$ is to consist of sets of the form $\omega(U \times (t_1, t_2))$, U open in E, $0 < t_1 < t_2 < 1$, or $\omega(p^{-1}(V) \times (t_1, 1])$, $t < 1$, or $\omega(p^{-1}(V) \times [0, t))$, $t > 0$, where V is open in B. Define

(7.3) $$\Sigma(p): \Sigma(E) \to B$$

by $\Sigma(p)([e, t]) = p(e)$.

(7.4) LEMMA. *If $\beta = (E, B, p)$ is locally trivial with fibre F, then $\Sigma(\beta) = (\Sigma(E), B, \Sigma(p))$ is locally trivial with fibre $S(F)$.*

This is easily checked. We will call $\Sigma(\beta)$ the *suspension* of β.

We will now describe a natural way of assigning to β a B-spectrum. Let $\Sigma^m(\beta) = \Sigma(\Sigma^{m-1}(\beta))$ and define

(7.5) $$\Delta_m: B \to \Sigma^m(E)$$

by $\Delta_m(b) = [e, 0 \cdots 0]$, $e \in p^{-1}(b)$. Note that Δ_m is a cross-section to $\Sigma^m(p): \Sigma^m(E) \to B$.

Let $\mathcal{S}(\beta)$ denote the B-spectrum consisting of pairs (Γ_m, δ_m) and maps

(7.6) $$(k_m, K_m): (\Gamma_m, \delta_m) \to (\Omega(\Gamma_{m+1}; \delta_{m+1}), \Omega(\delta_{m+1})),$$

where

(7.7) $$\begin{aligned} (\Gamma_m, \delta_m) &= (\Sigma^m(\beta), \Delta_m), \quad m > 0, \\ &= (\Omega^{-m+1}(\Sigma(\beta); \Delta_1), \Omega^{-m+1}(\Delta_1)), \quad m \leqq 0, \end{aligned}$$

and for $m > 0$

(7.8) $$k_m: \Sigma^m(E) \to \Omega(\Sigma^{m+1}(E); \Delta_{m+1})$$

is defined by

$$k_m([e, t_1, \ldots, t_m])(\lambda) = [\Delta_m p(e), 2\lambda], \quad 0 \leqq \lambda \leqq 1/2,$$
$$= [e, t_1, \ldots, t_m, 2-2\lambda], \quad 1/2 \leqq \lambda \leqq 1,$$

and

(7.9) $$K_m: B \times I \to \Omega(\Sigma^{m+1}(E), \Delta_{m+1})$$

is defined by

$$K_m(b, t)(\lambda) = [\Delta_m(b), 2t\lambda], \quad 0 \leqq \lambda \leqq 1/2,$$
$$= [\Delta_m(b), 2t(1-\lambda)], \quad 1/2 \leqq \lambda \leqq 1;$$

whereas for $m \leqq 0$,

(7.10) $$k_m: \Omega^{-m+1}(\Sigma(E); \Delta_1) \to \Omega^{-m+1}(\Sigma(E); \Delta_1)$$

is to be the identity, and

(7.11) $$K_m: B \times I \to \Omega^{-m+1}(\Sigma(E); \Delta_1)$$

is to be the constant homotopy $K_m(b, t) = \Omega^{-m+1}(\Delta_1)(b)$, $0 \leqq t \leqq 1$.

The *square* of $\beta = (E, B, p)$ is $\beta^2 = (E^2, E, p_1)$, where

(7.12) $$E^2 = \{(e_1, e_2) \in E \times E \mid p(e_1) = p(e_2)\}$$

and $p_1: E^2 \to E$ is given by $p_1(e_1, e_2) = e_1$. There is a cross-section $\Delta: E \to E^2$ by $\Delta(e) = (e, e)$. Now define

(7.13) $$\mu: E^2 \to \Omega(\Sigma(E); \Delta_1)$$

by

$$\mu(e_1, e_2)(\lambda) = [e_2, 2\lambda], \quad 0 \leqq \lambda \leqq 1/2,$$
$$= [e_1, 2-2\lambda], \quad 1/2 \leqq \lambda \leqq 1.$$

In the diagrams

(7.14). $$\begin{array}{ccc} E^2 & \xrightarrow{\mu} & \Omega(\Sigma(E); \Delta_1) \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \Omega(\Sigma(p))} \\ E & \xrightarrow{p} & B \end{array} \qquad \begin{array}{ccc} E^2 & \xrightarrow{\mu} & \Omega(\Sigma(E); \Delta_1) \\ \uparrow{\scriptstyle \Delta} & & \uparrow{\scriptstyle \Omega(\Sigma(\Delta_1))} \\ E & \xrightarrow{p} & B \end{array}$$

the first is commutative and the second is homotopy commutative, a connecting homotopy being

(7.15) $$M: E \times I \to \Omega(\Sigma(E); \Delta_1)$$

by

$$M(e, t)(\lambda) = [e, 2t\lambda], \quad 0 \leqq \lambda \leqq 1/2,$$
$$= [e, 2t(1-\lambda)], \quad 1/2 \leqq \lambda \leqq 1.$$

Note that M_t is a lifting of p for $0 \leqq t \leqq 1$.

Given $X \in \mathscr{P}^2$ and $f\colon X \to B$, let

$$(7.16) \qquad \mathscr{L}(X, f, \beta) = \{g\colon X \to E \mid pg = f\}$$

and

$$(7.17) \qquad L(X, f, \beta) = \pi_0(\mathscr{L}(X, f, \beta)).$$

Suppose that $L(X, f, \beta)$ is not empty. Let $\alpha \in L(X, f, \beta)$ be represented by $g\colon X \to E$ and define

$$(7.18) \qquad \psi_\alpha\colon L(X, f, \beta) \to h^0(X, f, \mathscr{S}(\beta))$$

to be the composition

$$(7.19) \quad L(X, f, \beta) \xrightarrow{\eta(g)} L(X, g, \beta^2) \xrightarrow{\mu_\#} L(X, f, \Omega(\Sigma(\beta); \Delta_1)) \longrightarrow h^0(X, f, \mathscr{S}(\beta)),$$

where $\eta(g)([q]) = [g \times q]$, $q \in \mathscr{L}(X, f, \beta)$, and the unmarked arrow is inclusion into the direct limit. Note that $\eta(g)$ is one-one and onto.

(7.20) Lemma. *The correspondence ψ_α is independent of the representative chosen for α.*

Proof. Let g' also represent α and let $H\colon X \times I \to E$ satisfy $H_0 = g$, $H_1 = g'$ and $pH_t = f$, $0 \leqq t \leqq 1$. Then, for $q \in \mathscr{L}(X, f, \beta)$, define

$$J\colon X \times I \to \Omega(\Sigma(E); \Delta_1)$$

by $J(x, t) = \mu(H(x, t), q(x))$, $x \in X$, $0 \leqq t \leqq 1$. We have $J_0 = \mu(g \times q)$, $J_1 = \mu(g' \times q)$ and $\Omega(\Sigma(p))J_t = f$, $0 \leqq t \leqq 1$. Therefore $[\mu(g \times q)] = [\mu(g' \times q)]$ in $L(X, f, \Omega(\Sigma(\beta); \Delta_1))$. This completes the proof.

We need now the following fact. Suppose that we have a commutative diagram

$$(7.21) \qquad \begin{array}{ccc} E_1 & \xrightarrow{\mu} & E_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow{\nu} & B_2 \end{array}$$

with both p_1 and p_2 fibre maps. Let F_1 and F_2 denote the respective fibres.

(7.22) Lemma. *Suppose that*

$$\mu_\#\colon \pi_m(F_1; e_1) \to \pi_m(F_2; e_2)$$

is an isomorphism, $m < 2n$. Then if $X \in \mathscr{P}^2$ is $(2n-1)$-coconnected, the correspondence

$$\mu_\#\colon L(X, f, \beta_1) \to L(X, \nu f, \beta_2), \quad f \in \mathscr{M}(X, B_1),$$

is one-one and onto.

This is well known.

(7.23) THEOREM. *Let $\beta=(E, B, p)$ be locally trivial with fibre F. If F is $(n-1)$-connected, $X \in \mathscr{P}^2$ is $(2n-1)$-coconnected and $L(X, f, \beta)$ is not empty, then ψ_α in* (7.18) *is one-one and onto.*

Proof. Apply the above lemma to conclude that both $\mu_\#$ in (7.19) and the inclusion of $L(X, f, \Omega(\Sigma(\beta); \Delta_1))$ into the direct limit $h^0(X, f, \mathscr{S}(\beta))$ are one-one and onto.

With X and β as in the above theorem, let $(L(X, f, \beta), \alpha)$ denote the set $L(X, f, \beta)$ together with the abelian group structure determined by the condition that ψ_α be an isomorphism. For $\gamma_1, \gamma_2 \in L(X, f, \beta)$, let $\gamma_1 +_\alpha \gamma_2$ denote their sum in $(L(X, f, \beta), \alpha)$. Using the homotopy M of (7.15), we see that α is the zero in this group.

(7.24) LEMMA. *For $\alpha_0, \alpha_1 \in L(X, f, \beta)$ we have $\psi_{\alpha_0}(\alpha_1) = -\psi_{\alpha_1}(\alpha_0)$.*

Proof. Let $g_0, g_1 \colon X \to E$ represent α_0, α_1 respectively. Then $\psi_{\alpha_0}(\alpha_1)$ is represented by $\mu(g_0 \times g_1)$ and $\psi_{\alpha_1}(\alpha_0)$ by $\mu(g_1 \times g_0)$. From the definition of μ, the product $\mu(g_0 \times g_1) \cdot \mu(g_1 \times g_0)$ is homotopic as a lifting of f to the trivial lifting $\Omega(\Delta_1)f$. That is, $\psi_{\alpha_0}(\alpha_1) + \psi_{\alpha_1}(\alpha_0) = 0$. This completes the proof.

(7.25) LEMMA. *For $\alpha_0, \alpha_1 \in L(X, f, \beta)$ and $v \in h^0(X, f, \mathscr{S}(\beta))$, we have $\psi_{\alpha_1}\psi_{\alpha_0}^{-1}(v) = \psi_{\alpha_1}(\alpha_0) + v$.*

Proof. Let $q \colon X \to E$ be a lifting of f such that $\mu(g_0 \times q)$ represents v. Then q represents $\psi_{\alpha_0}^{-1}(v)$ and $\psi_{\alpha_1}\psi_{\alpha_0}^{-1}(v)$ is represented by $\mu(g_1 \times q)$. From the definition of μ, we see that $\mu(g_1 \times q)$ is homotopic as a lifting of f to $\mu(g_1 \times g_0) \cdot \mu(g_0 \times q)$. The latter represents $\psi_{\alpha_1}(\alpha_0) + v$. This completes the proof.

(7.26) LEMMA. *For $\alpha_0, \alpha_1, \alpha_2, \gamma \in L(X, f, \beta)$, we have $\alpha_0 +_{\alpha_1} (\alpha_1 +_{\alpha_2} \gamma) = \alpha_0 +_{\alpha_2} \gamma$.*

Proof. By (7.24) and (7.25), we have

$$
\begin{aligned}
\psi_{\alpha_1}(\alpha_1 +_{\alpha_2} \gamma) &= \psi_{\alpha_1}\psi_{\alpha_2}^{-1}(\psi_{\alpha_2}(\alpha_1) + \psi_{\alpha_2}(\gamma)) \\
&= \psi_{\alpha_1}(\alpha_2) + \psi_{\alpha_2}(\alpha_1) + \psi_{\alpha_2}(\gamma) \\
&= \psi_{\alpha_2}(\gamma).
\end{aligned} \tag{7.27}
$$

Therefore

$$
\begin{aligned}
\alpha_0 +_{\alpha_1} (\alpha_1 +_{\alpha_2} \gamma) &= \psi_{\alpha_1}^{-1}(\psi_{\alpha_1}(\alpha_0) + \psi_{\alpha_1}(\alpha_1 +_{\alpha_2} \gamma)) \\
&= \psi_{\alpha_1}^{-1}(\psi_{\alpha_1}(\alpha_0) + \psi_{\alpha_2}(\gamma)) \\
&= \psi_{\alpha_1}^{-1}(\psi_{\alpha_1}\psi_{\alpha_0}^{-1}\psi_{\alpha_2}(\gamma)) = \psi_{\alpha_0}^{-1}\psi_{\alpha_2}(\gamma).
\end{aligned} \tag{7.28}
$$

On the other hand, by (7.25)

$$
\begin{aligned}
\alpha_0 +_{\alpha_2} \gamma &= \psi_{\alpha_2}^{-1}(\psi_{\alpha_2}(\alpha_0) + \psi_{\alpha_2}(\gamma)) \\
&= \psi_{\alpha_2}^{-1}(\psi_{\alpha_2}\psi_{\alpha_0}^{-1}\psi_{\alpha_2}(\gamma)) = \psi_{\alpha_0}^{-1}\psi_{\alpha_2}(\gamma).
\end{aligned} \tag{7.29}
$$

Comparing (7.28) and (7.29) gives the desired result.

Let $\mathscr{U}$ denote the category of sets and functions, let $\mathscr{C}$ be an arbitrary category and let $F \colon \mathscr{C} \to \mathscr{U}$ be a contravariant functor. We say that F has a *natural affine group structure* if for each object $X \in \mathscr{C}$ and element $\alpha \in F(X)$, there is a rule which assigns an abelian group structure to the set $F(X)$. Denote this group by $(F(X), \alpha)$ and for $\gamma_1, \gamma_2 \in F(X)$ let $\gamma_1 +_\alpha \gamma_2$ denote their sum in $(F(X), \alpha)$. The following conditions must be satisfied.

(A). The zero of $(F(X), \alpha)$ is α.

(B). If $g: X_1 \to X_2$ is a map in $\mathscr{C}$, then $F(g): (F(X_2), \alpha) \to (F(X_1), F(g)(\alpha))$ is a homomorphism.

(C). For $\alpha_0, \alpha_1 \in F(X)$, the translation $T(\alpha_0, \alpha_1): (F(X), \alpha_1) \to (F(X), \alpha_0)$ defined by $T(\alpha_0, \alpha_1)(\gamma) = \alpha_0 +_{\alpha_1} \gamma$ is an isomorphism.

(D). $T(\alpha_0, \alpha_0)$ is the identity, $\alpha_0 \in F(X)$.

(E). $T(\alpha_0, \alpha_1)T(\alpha_1, \alpha_2) = T(\alpha_0, \alpha_2)$, $\alpha_0, \alpha_1, \alpha_2 \in F(X)$.

Now let $\mathscr{P}(\beta, 2n-1)$ denote the category whose objects are pairs (X, f) with X a $(2n-1)$-coconnected CW-complex, $f \in \mathscr{M}(X, B)$ and $L(X, f, \beta)$ not empty. A map $g: (X_1, f_1) \to (X_2, f_2)$ in the category is to be $g: X_1 \to X_2$ such that $f_1 = f_2 g$.

(7.30) Theorem. *Let $\beta = (E, B, p)$ be locally trivial with fibre F which is $(n-1)$-connected. Then the set functor $L(X, f, \beta): \mathscr{P}(\beta, 2n-1) \to \mathscr{U}$ has a natural affine group structure.*

Proof. Properties (A) and (B) are easily checked. We will show now that

$$T(\alpha_0, \alpha_1): (L(X, f, \beta), \alpha_1) \to (L(X, f, \beta), \alpha_0)$$

is a homomorphism. By Lemma (7.26)

$$\begin{aligned} T(\alpha_0, \alpha_1)(\gamma_2 +_{\alpha_1} \gamma_2) &= \alpha_0 +_{\alpha_1} (\gamma_1 +_{\alpha_1} \gamma_2) \\ &= (\alpha_0 +_{\alpha_1} \gamma_1) +_{\alpha_0} (\alpha_0 +_{\alpha_1} \gamma_2) \\ &= T(\alpha_0, \alpha_1)(\gamma_1) +_{\alpha_0} T(\alpha_0, \alpha_1)(\gamma_2). \end{aligned}$$

Property (D) is evident and (E) is just Lemma (7.26). Finally (D) and (E) imply that $T(\alpha_0, \alpha_1)$ is an isomorphism.

8. **Equivariant maps.** Let G act as a group of transformations on X and Y. A map $f: X \to Y$ is *equivariant* if $f(gx) = gf(x)$, $g \in G$, $x \in X$. Two equivariant maps $f_0, f_1: X \to Y$ are *equivariantly homotopic* if there is a homotopy $F: X \times I \to Y$ from f_0 to f_1, such that F_t is equivariant, $0 \leqq t \leqq 1$. Let $E(X, Y)$ denote the set of equivariant homotopy classes of equivariant maps from X to Y. A map $f: X \to Y$ is an *equivariant homotopy equivalence* if there is $g: Y \to X$ such that fg and gf are equivariantly homotopic to the identity.

The following is an equivariant form of a theorem of J. H. C. Whitehead [10].

(8.1) Lemma. *Suppose that X and Y are connected CW-complexes on which the action of G is both free and cellular. If $f: X \to Y$ is equivariant and $f_\#: \pi_m(X; x_0) \to \pi_m(Y; y_0)$ is an isomorphism, $m \leqq \max(\dim X, \dim Y)$, then f is an equivariant homotopy equivalence.*

The proof can be carried out, using the mapping cylinder of f, along the same lines as the proof of the Whitehead theorem.

Let W be a G-free acyclic complex. For any space Z with an action of G, we have a locally trivial fibre space

$$\beta_z = (W\times Z/G,\ W/G,\ \pi_Z), \tag{8.2}$$

where G acts diagonally on $W\times Z$ and $\pi_Z\colon W\times Z/G\to W/G$ is induced by projection. The fibre is Z.

(8.3) Lemma. *Suppose X is a CW-complex on which the action of G is both free and cellular. Let $q\colon W\times X\to X$ be the projection. Then*

$$q^{\#}\colon E(W\times X,\ Y)\to E(X,\ Y)$$

is one-one and onto.

Proof. This follows from Lemma (8.1), since $q_{\#}\colon \pi_m(W\times X, (w_0, x_0))\to \pi_m(X; x_0)$ is an isomorphism, $m\geqq 0$.

Let X satisfy the hypothesis of the above lemma and let

$$\varphi\colon E(X,\ Y)\to L(W\times X/G,\ \pi_X,\ \beta_Y) \tag{8.4}$$

be the composition

$$E(X,\ Y)\xrightarrow{q^{\#-1}} E(W\times X,\ Y)\xrightarrow{\lambda} L(W\times X/G,\ \pi_X,\ \beta_Y),$$

where λ is defined as follows. Let $\alpha\in E(W\times X,\ Y)$ be represented by $g\colon W\times X\to Y$. We have $\tilde{g}\colon W\times X\to W\times Y$ by $\tilde{g}(w, x)=(w, g(w, x))$ and $\tilde{g}$ is equivariant. Its orbit map $\tilde{g}/G\colon W\times X/G\to W\times Y/G$ is a lifting of π_X. Let $\lambda(\alpha)$ be the class of $\tilde{g}/G$. The correspondence λ is essentially due to A. Heller [3] and is one-one and onto. Therefore φ is one-one and onto.

Fix a space Y and an action of G on Y. Let $\mathscr{Q}(Y, G, 2n-1)$ denote the category whose objects are CW-complexes X with an action of G which is both free and cellular and such that X/G is $(2n-1)$-coconnected and $E(X,\ Y)$ is not empty. The maps in the category are to be equivariant maps. We then have a covariant functor

$$D\colon \mathscr{Q}(Y, G, 2n-1)\to \mathscr{P}(\beta_Y, 2n-1) \tag{8.5}$$

which sends X to $(W\times X/G,\ \pi_X)$.

Suppose Y is $(n-1)$-connected. There are the set functors $E(\ \ ,Y)\colon \mathscr{Q}(Y, G, 2n-1)\to\mathscr{U}$ and $L(\ \ ,\beta_Y)\colon \mathscr{P}(\beta_Y, 2n-1)\to\mathscr{U}$ and φ in (8.4) is a natural transformation $E(\ \ ,Y)\to L(\ \ ,\beta_Y)D$. Since φ is one-one and onto, we may, for $X\in\mathscr{Q}(Y, G, 2n-1)$ and $\alpha\in E(X,\ Y)$, define an abelian group $(E(X,\ Y),\alpha)$ with underlying set $E(X,\ Y)$ by the condition that

$$\varphi\colon (E(X,\ Y),\alpha)\to (L(W\times X/G,\ \pi_X,\ \beta_Y),\ \varphi(\alpha)) \tag{8.6}$$

be an isomorphism. Then from Theorem (7.30) we have

(8.7) Theorem. *Let Y be an $(n-1)$-connected space with an action of the finite group G on Y. Then the set functor $E(\ \ ,Y)\colon \mathscr{Q}(G, 2n-1)\to\mathscr{U}$ has a natural affine group structure.*

REMARK. The addition in $(E(X, Y), \alpha)$ has a very simple description. Let $g, k_1, k_2\colon X \to Y$ represent $\alpha, \gamma_1, \gamma_2$ respectively. Let G act diagonally on Y^3 and consider the equivariant map $g \times k_1 \times k_2\colon X \to Y^3$. The subspace

$$V(Y) = \{(y_1, y_2, y_3) \in Y^3 \mid y_1 = y_2 \text{ or } y_1 = y_3\}$$

is invariant and $\pi_m(Y^3, V(Y)) = 0$, $m \leqq 2n-1$. Since X/G is $(2n-1)$-coconnected we may construct a homotopy $H\colon X \times I \to Y^3$ such that $H_0 = g \times k_1 \times k_2$, $H_1(X) \subset V(Y)$ and H_t is equivariant, $0 \leqq t \leqq 1$. Define a folding map $\lambda\colon V(Y) \to Y$ by $\lambda(y, y_2, y) = y_2$ and $\lambda(y, y, y_3) = y_3$. A representative of $\gamma_1 +_\alpha \gamma_2$ is λH_1. We will not need this fact so we will not stop to prove it.

Let $[X, Y]$ denote the track group of homotopy classes of maps from X to Y and let $\zeta\colon E(X, Y) \to [X, Y]$ assign to an equivariant homotopy class its ordinary homotopy class. Define

$$\theta\colon (E(X, Y), \alpha) \to [X, Y] \tag{8.8}$$

by $\theta(\gamma) = \zeta(\gamma) - \zeta(\alpha)$.

Fix base-points $w_0 \in W$ and $y_0 \in Y$. Let $i\colon X \to W \times X/G$ be given by $i(x) = [w_0, x]$, $x \in X$. This identifies X with the fibre $\pi_X^{-1}([w_0])$. Let $w_0^*\colon X \to W/G$ and $(w_0, y_0)^*\colon X \to W \times Y/G$ be the constant maps at $[w_0]$ and $[w_0, y_0]$ respectively and let $z \in L(X, w_0^*, \beta_Y)$ be the class of $(w_0, y_0)^*$. Consider the diagram

$$\begin{array}{ccccc} [X, Y] & \xrightarrow{\varphi_0} & L(X, w_0^*, \beta_Y) & \xrightarrow{\psi_z} & h^0(X, w_0^*, \mathscr{S}(\beta_Y)) \\ \uparrow{\scriptstyle\theta} & & & & \uparrow{\scriptstyle i^*} \\ (E(X, Y), \alpha) & \xrightarrow{\varphi} & L(W \times X/G, \pi_X, \beta_Y) & \xrightarrow{\psi_{\varphi(\alpha)}} & h^0(W \times X/G, \pi_X, \mathscr{S}(\beta_Y)), \end{array} \tag{8.9}$$

where φ_0 is defined as follows. Given $g\colon X \to Y$ define $g_0\colon X \to W \times Y/G$ by $g_0(x) = [w_0, g(x)]$, $x \in X$. Then let $\varphi_0(g) = [g_0]$.

We have an operation

$$\rho\colon G \times [X, Y] \to [X, Y] \tag{8.10}$$

of G on $[X, Y]$ defined by $\rho(g, \gamma) = (g^{-1})^{\#} g_{\#}(\gamma)$, $g \in G$, $\gamma \in [X, Y]$.

Next, we have a fibre map $\pi_X\colon W \times X/G \to W/G$ with fibre X. Take $f \in \mathscr{M}(W/G, W/G)$ to be the identity. Then from §4, the collection $h^0(\pi_X^{-1}([w]), \pi_X, \mathscr{S}(\beta_Y))$, $[w] \in W/G$, is a local system over W/G. Out of this we obtain an operation

$$\bar\rho\colon \pi_1(W/G; [w_0]) \times h^0(X, w_0^*, \mathscr{S}(\beta_Y)) \to h^0(X, w_0^*, \mathscr{S}(\beta_Y)). \tag{8.11}$$

Make the canonical identification of G with $\pi_1(W/G; [w_0])$.

(8.12) LEMMA. *The diagram* (8.9) *is commutative and*

$$\psi_z \varphi_0\colon [X, Y] \to h^0(X, w_0^*, \mathscr{S}(\beta_Y))$$

is an operator isomorphism.

The proof is tedious but straightforward and will be omitted. As a consequence of the lemma, θ is a homomorphism.

For a group A with G as left operators, let $I(A)$ denote its subgroup of invariant elements. Note that the image of θ is contained in $I([X, Y])$. For an integer n, let $\mathscr{A}(n)$ denote the class of abelian torsion groups whose p-primary component is zero if p does not divide n.

(8.13) THEOREM. *Let G have order n. Then*

$$\theta: (E(X, Y), \alpha) \to I([X, Y])$$

is an isomorphism modulo $\mathscr{A}(n)$.

Proof. By the preceding lemma it is sufficient to show that

$$i^*: h^0(W\times X/G, \pi_X, \mathscr{S}(\beta_Y)) \to I(h^0(X, w_0^*, \mathscr{S}(\beta_Y))) \tag{8.14}$$

is an isomorphism modulo $\mathscr{A}(n)$. Applying the spectral sequence of §4 to π_X: $W\times X/G \to W/G$, we have

$$E_2^{p,q} = H^p(W/G; [h^q(X, \mathscr{S}(\beta_Y))])$$

and a finite filtration

$$h^0(W\times X/G, \pi_X, \mathscr{S}(\beta_Y)) = J^{0,0} \supset J^{1,-1} \supset \dots \supset J^{k,-k} \supset \dots$$

with $E_\infty^{k,-k} = E_r^{k,-k}$ for large r.

We need the well-known facts [6] that $H^p(W/G; [h^q(X, \mathscr{S}(\beta_Y))])$ is in $\mathscr{A}(n)$, $p>0$, and

$$H^0(W/G; [h^q(X, \mathscr{S}(\beta_Y))]) \simeq I(h^q(X, w_0^*, \mathscr{S}(\beta_Y))).$$

From the above filtration we have that i^* in (8.14) is an isomorphism modulo $\mathscr{A}(n)$. This completes the proof.

Suppose $G=Z_2$ and $Y=S^n$, where the action of Z_2 on S^n is given by the antipodal map. Let $\Sigma^n(X)$ denote the nth stable cohomotopy group of X.

(8.15) COROLLARY. *Let T be a cellular fixed point free involution on the CW-complex X with X/T $(2n-1)$-coconnected. Then*

$$\theta: (E(X, S^n), \alpha) \to I(\Sigma^n(X))$$

is an isomorphism modulo 2-torsion.

Let $\omega: \Sigma^n(X) \to H^n(X)$ be the Hopf map and let Q denote the rational numbers. A theorem of Serre [8] asserts that

$$\omega \otimes 1: \Sigma^n(X) \otimes Q \to H^n(X; Q) \tag{8.16}$$

is an isomorphism.

Let Z_2 operate on $H^n(X; Q)$ by the rule $U \to T^*(u)$, n-odd, and $U \to -T^*(u)$, n-even.

(8.17) COROLLARY. *With X and T as in* (8.15)

$$\omega\theta \otimes 1: (E(X, S^n), \alpha) \otimes Q \to I(H^n(X; Q))$$

is an isomorphism.

Proof. Note that with the operation defined above on $H^n(X; Q)$, $\omega \otimes 1$ is an operator isomorphism. Now apply (8.15).

9. **Immersions and embeddings.** For a closed C^∞-manifold M of dimension n let $T(M)$ and $T_0(M)$ denote respectively its tangent bundle and tangent sphere bundle. Let E^{n+k} denote Euclidean $(n+k)$-space. An *immersion* $f: M \to E^{n+k}$ is a C^∞-map whose derivative $T(f): T(M) \to T(E^{n+k})$ has rank n at each point $x \in M$. Two immersions $f_0, f_1: M \to E^{n+k}$ are *regularly homotopic* if there is a C^∞-map $F: M \times I \to E^{n+k}$ such that $F_0 = f_0$, $F_1 = f_1$, and F_t is an immersion, $0 \leqq t \leqq 1$. Let $IM^{n+k}(M)$ denote the set of regular homotopy classes of immersions of M into E^{n+k}.

An *embedding* $f: M \to E^{n+k}$ is a one-one immersion. Two embeddings $f_0, f_1: M \to E^{n+k}$ are *isotopic* if there is a C^∞-map $F: M \times I \to E^{n+k}$ such that $F_0 = f_0$, $F_1 = f_1$ and F_t is an embedding, $0 \leqq t \leqq 1$. Let $EM^{n+k}(M)$ denote the set of isotopy classes of embeddings of M into E^{n+k}.

There is a fixed point free involution A_M on $T_0(M)$ which on each fibre S^{n-1} is the antipodal map A_{m-1}.

Let Δ denote the diagonal of $M \times M$. There is a fixed point free involution B_M on $M \times M - \Delta$ defined by $(x, y) \to (y, x)$.

An immersion $f: M \to E^{n+k}$ determines an equivariant map $T_0(f): T_0(M) \to E^{n+k} \times S^{n+k-1}$. Since the projection $\pi: E^{n+k} \times S^{n+k-1} \to S^{n+k-1}$ is equivariant, so also is $\pi T_0(f): T_0(M) \to S^{n+k-1}$.

An embedding $f: M \to E^{n+k}$ gives an equivariant map $f \times f: M \times M - \Delta \to E^{n+k} \times E^{n+k} - \Delta$. There is $\lambda: E^{n+k} \times E^{n+k} - \Delta \to S^{n+k-1}$ by $\lambda(v_1, v_2) = v_1 - v_2/|v_1 - v_2|$ and λ is equivariant. Then $\lambda(f \times f): M \times M - \Delta \to S^{n+k-1}$ is also equivariant.

Our study of the sets $IM^{n+k}(M)$ and $EM^{n+k}(M)$ is based on the following

(9.1) THEOREM (HIRSCH-HAEFLIGER [5]). *Suppose* $2k > n+1$. *The correspondence*

$$\eta: IM^{n+k}(M) \to E(T_0(M); S^{n+k-1})$$

defined by $\eta([f]) = [\pi T_0(f)]$ *is one-one and onto.*

(9.2) THEOREM (HAEFLIGER [4]). *Suppose* $2k > n+3$. *The correspondence*

$$\tau: EM^{n+k}(M) \to E(M \times M - \Delta; S^{n+k-1})$$

defined by $\tau([f]) = [\lambda(f \times f)]$ *is one-one and onto.*

By means of η and τ the sets $IM^{n+k}(M)$ and $EM^{n+k}(M)$ inherit a natural affine group structure.

(9.3) THEOREM. *For $k>1$,*

$$A_M^{\#}\colon \Sigma^{n+k-1}(T_0(M)) \to \Sigma^{n+k-1}(T_0(M))$$

is $(-1)^n$ times the identity, modulo 2-torsion.

Proof. There is a spectral sequence $\{E_r\}$ with

$$E_2^{p,q} = H^p(M; [\Sigma^q(S^{n-1})])$$

and a filtration

$$\Sigma^{n+k-1}(T_0(M)) = J^{0,n+k-1} \supset \cdots \supset J^{n,k-1} \supset 0$$

with $J^{p,q}/J^{p+1,q-1} = E_\infty^{p,q}$. It is sufficient to show that for $q>0$, the induced automorphism of $E_2^{p,q}$ is $(-1)^n$ times the identity. This agrees with the coefficient automorphism determined by $A_{m-1}^{\#}\colon \Sigma^q(S^{n-1}) \to \Sigma^q(S^{n-1})$, since A_{m-1} is the restriction of A_M to the fibre. It is well known that for $q>0$, $A_{m-1}^{\#}$ is $(-1)^n$ times the identity. This completes the proof.

Letting $I(\Sigma^{n+k-1}(T_0(M)))$ denote the subgroup of elements invariant under $(A_{m+k-1})_{\#}A_M^{\#}$, we have by the above lemma

(9.4) COROLLARY. *Let $k>1$. For k even, $I(\Sigma^{n+k-1}(T_0(M))) = \Sigma^{n+k-1}(T_0(M))$ and for k odd, $I(\Sigma^{n+k-1}(T_0(M)))=0$, modulo 2-torsion.*

We will write $M \subseteq E^{n+k}$ ($M \subset E^{n+k}$) if there exists an immersion (embedding) of M in E^{n+k}. Applying (8.15) and the preceding corollary, we have

(9.5) THEOREM. *Suppose $2k>n+1$ and $M \subseteq E^{n+k}$. For k-odd $(IM^{n+k}(M), \alpha)=0$ modulo 2-torsion. For k-even*

$$\theta_\eta\colon (IM^{n+k}(M), \alpha) \to \Sigma^{n+k-1}(T_0(M))$$

is an isomorphism modulo 2-torsion.

For embeddings we have

(9.6) THEOREM. *Suppose $2k>n+3$ and $M \subset E^{n+k}$. Then*

$$\theta_T\colon (EM^{n+k}(M), \alpha) \to I(\Sigma^{n+k-1}(M \times M - \Delta))$$

is an isomorphism modulo 2-torsion.

Here $I(\Sigma^{n+k-1}(M\times M-\Delta))$ is the subgroup of elements invariant under $(A_{m+k-1})_{\#}B_M^{\#}$.

10. **Rank of $IM^{n+k}(M)$ and $EM^{n+k}(M)$.** In this section it is assumed that M is orientable. Let

$$\tilde{\omega}\colon (IM^{n+k}(M), \alpha) \to H^k(M) \tag{10.1}$$

be the composition

$$(IM^{n+k}(M), \alpha) \xrightarrow{\theta_\eta} \Sigma^{n+k-1}(T_0(M)) \xrightarrow{\omega} H^{n+k-1}(T_0(M)) \xrightarrow{\psi} H^k(M)$$

where ψ is from the Gysin sequence for $T_0(M) \to M$.

For an immersion $g: M \to E^{n+k}$, the *normal class* of g is the Euler class $\chi(g) \in H^k(M)$ of the normal bundle of g.

(10.2) LEMMA. *Let γ generate $H^{n+k-1}(S^{n+k-1})$. Then with a suitable orientation of the normal bundle of g, we have*

$$\chi(g) = \psi T_0(g)^* \pi^*(\gamma).$$

Proof. This follows from Theorem (1.1) of [7]. Let $g^{-1}(T_0(E^{n+k}))$ be the bundle over M induced by g and let

$$T_0(M) \xrightarrow{f_2} g^{-1}(T_0(E^{n+k})) \xrightarrow{\tilde{g}} T_0(E^{n+k})$$

be the factorization of $T_0(g)$. Then

$$\psi T_0(g)^* \pi^*(\gamma) = \psi f_2^* \tilde{g}^* \pi^*(\gamma). \tag{10.3}$$

In the Gysin sequence for $g^{-1}(T_0(E^{n+k})) \to M$, we have $\psi \tilde{g}^* \pi^*(\gamma) = 1 \in H^0(M)$. Using the notation of Theorem (1.1) of [7], we have

$$\psi f_2^* \tilde{g}^* \pi^*(\gamma) = G_2 \psi \tilde{g}^* \pi^*(\gamma) = G_2^*(1) = \chi(g).$$

This completes the proof.

The above lemma implies that $\chi(g)$ depends only on the class $\beta \in IM^{n+k}(M)$, so we will write $\chi(\beta)$ instead of $\chi(g)$.

(10.4) LEMMA. *For $\gamma \in IM^{n+k}(M)$, $\tilde{\omega}(\gamma) = \chi(\gamma) - \chi(\alpha)$.*

Proof. This follows from the preceding lemma and the definition of $\tilde{\omega}$.

Now from Theorem (9.5) we have

(10.5) THEOREM. *Suppose $2k > n+1$ and $M \subseteq E^{n+k}$. For k-odd, $(IM^{n+k}(M), \alpha) \otimes Q = 0$ and for k-even*

$$\tilde{\omega} \otimes 1: (IM^{n+k}(M), \alpha) \otimes Q \to H^k(M; Q),$$

given by $\tilde{\omega} \otimes 1(\gamma \otimes x) = (\chi(\gamma) - \chi(\alpha)) \otimes x$, is an isomorphism.

For embeddings, let

$$\tilde{\omega}: (EM^{n+k}(M), \alpha) \to H^{n+k-1}(M \times M - \Delta) \tag{10.6}$$

be the composition

$$(EM^{n+k}(M), \alpha) \xrightarrow{\theta_T} \Sigma^{n+k-1}(M \times M - \Delta) \xrightarrow{\omega} H^{n+k-1}(M \times M - \Delta).$$

By Theorem (8.15) we have an isomorphism

$$\tilde{\omega} \otimes 1: (EM^{n+k}(M), \alpha) \otimes Q \to I(H^{n+k-1}(M \times M - \Delta; Q)). \tag{10.7}$$

Let $u \in H_{2n}(M\times M)$ be a fundamental class and let

(10.8) $$D\colon H^{n+k-1}(M\times M-\Delta; Q) \to H_{n-k+1}(M\times M, \Delta; Q)$$

denote the Lefschetz-Poincaré duality map

$$D(v) = u \cap v, \qquad v \in H^{n+k-1}(M\times M-\Delta; Q).$$

Next, let

(10.9) $$\kappa\colon H_*(M; Q) \otimes H_*(M; Q) \to H_*(M\times M; Q)$$

be the Künneth map and $\delta\colon M \to M\times M$ the diagonal map. For $a \in H_{n-k+1}(M)$, write

$$\kappa^{-1}\delta_*(a) = a \otimes 1 \oplus 1 \otimes a + \hat{a},$$

$$\hat{a} \in \sum_{i=1}^{n-k} H_i(M; Q) \otimes H_{i'}(M; Q),$$

$i' = n-k+1-i$. The element a is *primitive* if $\hat{a}=0$. Let $j\colon M\times M \to (M\times M, \Delta)$ be the inclusion. We have an isomorphism

(10.10) $$(j_*\kappa)^{-1}D\colon H^{n+k-1}(M\times M-\Delta; Q) \to \sum_{i=1}^{n-k+1} H_i(M; Q) \otimes H_{i'}(M; Q).$$

Define an involution T on the right hand side of (10.10) by

(10.11) $$\begin{aligned} T(a \otimes 1) &= (-a) \otimes 1 - \hat{a}, && \dim(a) = n-k+1,\\ T(a \otimes b) &= (-1)^{ii'} b \otimes a, && \dim(a) = i, \dim(b) = i' > 0. \end{aligned}$$

Then $(j_*\kappa)^{-1}D$ is an operator isomorphism when the involution on the left is $(-1)^{n+k}B_M^*$ and on the right is $(-1)^k T$. Now let

(10.12) $$J_{n,k}(M; Q) = \sum_{i=1}^{r} H_i(M; Q) \otimes H_{i'}(M; Q),$$

where r is the greatest integer less than or equal to $n-k/2$ and let $P_{n-k+1}(M; Q)$ denote the subgroup of primitive elements in $H_{n-k+1}(M; Q)$.

(10.13) Theorem *Suppose $2k > n+3$ and $M \subset E^{n+k}$. Then $(EM^{n+k}(M), \alpha) \otimes Q$ is given by the following table*:

	$k \equiv 0 \bmod 2$	$k \equiv 1 \bmod 2$
$n+k \equiv 0, 1, 2 \bmod 4$	$J_{n,k}(M; Q)$	$J_{n,k}(M; Q) \oplus P_{n-k+1}(M; Q)$
$n+k \equiv 3 \bmod 4$	$J_{n,k}(M; Q) \oplus H_{n-k+1/2}(M; Q)$	$J_{n,k}(M; Q) \oplus H_{n-k+1/2}(M; Q) \oplus P_{n-k+1}(M; Q)$

Proof. The various cases are all handled in the same way. For example, when $n+k \equiv 0, 1, 2 \bmod 4$ and $k \equiv 0 \bmod 2$,

$$\rho: J_{n,k}(M; Q) \to \sum_{i=1}^{n-k+1} H_i(M; Q) \otimes H_{i'}(M; Q)$$

by

$$\rho\left(\sum_{i=1}^{r} a_i \otimes b_{i'}\right) = \sum_{i=1}^{r} (a_i \otimes b_{i'} \oplus (-1)^{ii'} b_{i'} \otimes a_i)$$

is injective onto the subgroup of elements invariant under $(-1)^k T$.

11. **The normal class of an immersion.** In this section, M is orientable. We consider a question raised by Lashof and Smale [7] as to what elements $v \in H^k(M)$ are realizable as normal classes of an immersion. Such elements will be characterized as permanent cycles in a spectral sequence.

If $M \subseteq E^{n+k}$ let

(11.1) $$N^k(M) = \{v \in H^k(m) \mid v = \chi(\gamma), \gamma \in IM^{n+k}(M)\}.$$

If $2k > n+1$, then by Lemma (10.4), $N^k(M)$ is a coset of $\tilde{\omega}(IM^{n+k}(M), \alpha)$. If it is assumed that $M \subseteq E^{n+k-1}$ or $M \subset E^{n+k}$, then there is $\alpha \in IM^{n+k}(M)$ such that $\chi(\alpha) = 0$. In this case $N^k(M) = \tilde{\omega}(IM^{n+k}(M), \alpha)$ and is therefore a subgroup of $H^k(M)$.

Let S^∞ and P^∞ be the infinite dimensional sphere and projective space respectively, let $X(M) = S^\infty \times T_0(M)/Z_2$ and $P_0(M) = T_0(M)/Z_2$ and let $\pi_1: X(M) \to P^\infty$ and $\pi_2: X(M) \to P_0(M)$ be the projections. Pick $s_0 \in S^\infty$ and define $i: T_0(M) \to X(M)$ by $i(x) = [s_0, x]$, $x \in T_0(M)$. From the definition of $\tilde{\omega}$ and the commutativity of (8.9) we see that the image of $\tilde{\omega}$ is equal to the image of the composition

(11.2) $$h^0(X(M), \pi_1, \mathcal{S}(\beta)) \xrightarrow{i^*} \Sigma^{n+k-1}(T_0(M)) \xrightarrow{\omega} H^{n+k-1}(T_0(M)) \xrightarrow{\psi} H^k(M),$$

where $\beta = (S^\infty \times S^{n+k-1}/Z_2, P^\infty, \pi_1)$.

Constructing the spectral sequence for the identity map $X(M) \to X(M)$ and $\pi_1 \in \mathcal{M}(X(M), P^\infty)$, we have

(11.3) $$E_2^{p,q} = H^p(X(M); [\Sigma^{q+n+k-1}(pt)]) = H^p(P_0(M); [\Sigma^{q+n+k-1}(pt)])$$

(the right-hand identification being made by π_2^*) and

(11.4) $$j: h^0(X(M), \pi_1, \mathcal{S}(\beta)) \to E_\infty^{n+k-1, -(n+k-1)} \subset H^{n+k-1}(P_0(M); [Z]).$$

Then the following diagram is commutative.

(11.5)
$$\begin{tikzcd} h^0(X(M), \pi_1, \mathcal{S}(\beta)) \arrow[r, "i^*"] \arrow[d, "j"] & \Sigma^{n+k-1}(T_0(M)) \arrow[d, "\omega"] \\ H^{n+k-1}(P_0(M); [Z]) \arrow[r, "p^*"] & H^{n+k-1}(T_0(M)) \end{tikzcd}$$

where $p: T_0(M) \to P_0(M)$ is the orbit map. Combining the above facts, we have

(11.6) THEOREM. *Suppose $2k > n+1$ and $M \subset E^{n+k}$ or $M \subseteq E^{n+k-1}$. Then*

$$N^k(M) = \psi p^*(E_\infty^{n+k-1, -(n+k-1)}).$$

(11.7) LEMMA. *For k-odd $\psi p^*(H^{n+k-1}(P_0(M); [Z])) = 0$. For k-even, $\psi p^*(H^{n+k-1}(P_0(M); [Z])) \subset 2H^k(M)$.*

Proof. Comparing the spectral sequence for $T_0(M) \to M$ and $P_0(M) \to M$, we have a commutative diagram

$$\begin{tikzcd}
H^{n+k-1}(P_0(M); [Z]) \arrow[r, "p^*"] \arrow[d, "j"] & H^{n+k-1}(T_0(M)) \arrow[d, "j"] \arrow[ddr, "\psi"] & \\
E_\infty^{k,n-1} \arrow[d, phantom, "\cap"] & E_\infty^{k,n-1} \arrow[d, equal] & \\
H^k(M; H^{n-1}(P^{n-1}; [Z])) \arrow[r, "(\tilde{p}^*)_\#"] & H^k(M; H^{n-1}(S^{n-1})) \arrow[r, phantom, "="] & H^k(M),
\end{tikzcd}$$

where $\tilde{p}: S^{n-1} \to P^{n-1}$ is the restriction of p to the fibre. The involution on Z is $(-1)^{n+k}$. Thus $H^{n-1}(P^{n-1}; [Z])$ is Z_2 or Z depending on whether k is odd or even. In the former case $\tilde{p}^*$ has image 0 and in the latter $\tilde{p}^*$ has image $2H^{n-1}(S^{n-1})$. The lemma now follows from the commutativity of the above diagram.

From (11.6), (11.7) and Theorem (10.6), we have

(11.8) THEOREM. *Suppose $2k > n+1$ and $M \subseteq E^{n+k-1}$. For k-odd, $N^k(M) = 0$. For k-even, $N^k(M)$ is a subgroup of $2H^k(M)$ having finite index.*

REMARK. For $k = n$ or $n-1$ and k-even, we deduce that

$$\psi p^*(H^{n+k-1}(P_0(M); [Z])) = 2H^k(M).$$

Then using Theorem (11.6), we obtain the following table for $N^k(M)$:

	n-even	n-odd
$k=n$	$2H^n(M)$	0
$k=n-1$	0	$2H^{n-1}(M)$

The values for $k = n$ were given by Lashof and Smale [7]. Information for $k < n-1$ would involve computing the twisted cohomology operations which appear as differential operators in the spectral sequence.

Let $S^k \times S^{n-1}$ have the involution $(a, b) \to (a, -b)$.

(11.9) LEMMA. *For k-even, there is an equivariant map $f: S^k \times S^{n-1} \to S^{k+n-1}$ of degree 2.*

Proof. Let $p: S^k \times S^{n-1} \to S^k \times P^{n-1}$ be the orbit map. There is an equivariant map $g: S^k \times S^{n-1} \to S^{k+n-1}$ of degree 0, namely, the projection onto S^{n-1} followed by the inclusion $S^{n-1} \subset S^{k+n-1}$. Now the correspondence

$$E(\tilde{S}^k \times S^{n-1}, S^{k+n-1}) \to H^{k+n-1}(S^k \times P^{n-1}; [Z])$$

which assigns to f the primary obstruction $d(f, g)$ to an equivariant homotopy between f and g is one-one and onto. The involution on Z is $(-1)^{k+n} = (-1)^n$ and, by an elementary computation, $H^{k+n-1}(S^k \times P^{n-1}; [Z]) = Z$ and

$$p^*: H^{k+n-1}(S^k \times P^{n-1}; [Z]) \to H^{k+n-1}(S^k \times S^{n-1})$$

takes a generator to twice a generator. Therefore if we choose f so that $d(f, g)$ is a generator, the degree of f will be 2.

Suppose that M is parallelizable. Then $T_0(M) = M \times S^{n-1}$ and we can define

$$\psi: \Sigma^{n+k-1}(T_0(M)) \to \Sigma^k(M) \tag{11.10}$$

to be the composition

$$\Sigma^{n+k-1}(M \times S^{n-1}) \xrightarrow{j^{*-1}} \Sigma^{n+k-1}(M \times S^{n-1}, M \times S^{n-1}) \longrightarrow \Sigma^k(M),$$

where j is inclusion and the unmarked arrow is $(n-1)$-fold desuspension. The following diagram is commutative:

$$\begin{array}{ccc} \Sigma^{n+k-1}(T_0(M)) & \xrightarrow{\psi} & \Sigma^k(M) \\ \big\downarrow \omega & & \big\downarrow \omega \\ H^{n+k-1}(T_0(M)) & \xrightarrow{\psi} & H^k(M) \end{array} \tag{11.11}$$

(11.12) Theorem. *Suppose M is parallelizable and $2k > n+1$. Then*

$$\psi\theta\eta: (IM^{n+k}(M), \alpha) \to \Sigma^k(M)$$

has image 0, *k-odd, and* $2\Sigma^k(M)$, *k-even.*

Proof. We will first show that the image of

$$i^*: h^0(X(M), \pi_1, \mathcal{S}(\beta)) \to \Sigma^{n+k-1}(T_0(M))$$

is 0, k-odd, and is contained in $2\Sigma^{n+k-1}(T_0(M))$, k-even. This will imply that the image of $\psi\theta\eta$ is 0, k-odd, and contained in $2\Sigma^k(N)$, k-even. We have a commutative diagram

$$\begin{array}{ccc} T_0(M) & \xrightarrow{i} & X(M) \\ \big\downarrow \tilde{p} & & \big\downarrow p \\ S^{n-1} & \xrightarrow{\tilde{\imath}} & S^\infty \times S^{n-1}/Z_2 \end{array}$$

where ρ and $\tilde{\rho}$ are projections and $\tilde{\imath}$ is inclusion. Comparing the spectral sequences for ρ and $\tilde{\rho}$, we obtain a commutative diagram

$$\begin{tikzcd} h^0(X(M), \pi_1, \mathscr{S}(\beta)) \arrow[r, "i^*"] \arrow[d, "j"] & \Sigma^{n+k-1}(T_0(M)) \arrow[d, "\tilde{j}"] \\ H^{n-1}(S^\infty \times S^{n-1}/Z_2; [\Sigma^k(M)]) \arrow[r, "\tilde{\imath}^*"] & H^{n-1}(S^{n-1}; \Sigma^k(M)) \end{tikzcd}$$

Since $\tilde{j}$ is an isomorphism and $\tilde{\imath}^*$ has image 0, k-odd, and $2H^{n-1}(S^{n-1}; \Sigma^k(M))$, k-even, it follows that the image of i^* is 0, k-odd, and contained in $2\Sigma^{n+k-1}(T_0(M))$, k-even.

Suppose now that k is even. Let $u \in 2\Sigma^k(M)$ and choose $f\colon M \to S^k$ such that $u = 2[f]$. Let $f'\colon M \times S^{n-1} \to S^k \times S^{n-1}$ be defined by $f'(x, b) = (f(x), b)$. Then f' is equivariant when the involution on $M \times S^{n-1}$ is $(x, b) \to (x, -b)$ and on $S^k \times S^{n-1}$ is $(a, b) \to (a, -b)$. By Lemma (11.9), there is an equivariant map $g\colon S^k \times S^{n-1} \to S^{n+k-1}$ of degree 2. Let $\gamma \in E(T_0(M), S^{n+k-1})$ be the class of gf'. Then choosing $\alpha \in IM^{n+k}(M)$ so that $\zeta\eta(\alpha) = 0$ in $\Sigma^{n+k-1}(T_0(M))$, (see (8.8)), we have

$$\psi\theta\eta(\eta^{-1}(\gamma)) = 2[f] = u.$$

Therefore, when k is even, the image of $\psi\theta\eta$ is onto $2\Sigma^k(M)$. This completes the proof.

Let $S^k(M)$ denote the subgroup of spherical classes of $H^k(M)$, that is, the image of $\omega\colon \Sigma^k(M) \to H^k(M)$.

(11.13) COROLLARY. *Suppose M is parallelizable and $2k > n+1$. Then $N^k(M) = 0$, k-odd, and $N^k(M) = 2S^k(M)$, k-even.*

Proof. Choose $\alpha \in IM^{n+k}(M)$ such that $\zeta\eta(\alpha) = 0$. Suppose $v \in N^k(M)$. Let $\gamma \in IM^{n+k}(M)$ be such that $v = \chi(\gamma)$. Then by Lemma (10.4), $v = \tilde{\omega}(\gamma)$. By the preceding theorem, there is $u \in \Sigma^k(M)$ such that $\psi\theta\eta(\gamma) = 2u$. Then, by the commutativity of (11.11),

$$v = \tilde{\omega}(\gamma) = 2\omega(u) \in 2S^k(M).$$

Conversely, suppose $v \in 2S^k(M)$. Let $u \in \Sigma^k(M)$ be such that $v = 2\omega(u)$. By the preceding theorem there is $\gamma \in IM^{n+k}(M)$ such that $\psi\theta\eta(\gamma) = 2u$. By the commutativity of (9.12)

$$\chi(\gamma) = \tilde{\omega}(\gamma) = \omega(2u) = v.$$

This completes the proof.

REMARK. Theorems (11.12) and (11.13) are also true if M is a π-manifold. The proofs are essentially the same.

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